

Annular Vortex Solutions to the Landau-Ginzburg Equations in Mesoscopic Superconductors

Jan Govaerts, Geoffrey Stenuit, Damien Bertrand and Olivier van der Aa¹

*Institut de Physique Nucléaire
Université catholique de Louvain
2, Chemin du Cyclotron
B-1348 Louvain-la-Neuve, Belgium*

Abstract

New vortex solutions to the Landau-Ginzburg equations are described. These configurations, which extend the well known Abrikosov and giant magnetic vortex ones, consist of a succession of ring-like supercurrent vortices organised in a concentric pattern, possibly bound to a giant magnetic vortex then lying at their center. The dynamical and thermodynamic stability of these annular vortices is an important open issue on which hinges the direct experimental observation of such configurations. Nevertheless, annular vortices should affect indirectly specific dynamic properties of mesoscopic superconducting devices amenable to physical observation.

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¹E-mail addresses: govaerts, stenuit, bertrand, vanderaa @fynu.ucl.ac.be

1. Introduction. Ever since they were introduced, the Landau-Ginzburg (LG) equations[1] have remained a powerful and insightful tool for the exploration and understanding of superconducting and other collective quantum phenomena, in particular in low temperature superconductors. Among a host of important results and developments which were achieved through the use of these equations, the existence of Abrikosov magnetic vortex solutions[1] still stands out as a most remarkable prediction transcending the boundaries of condensed matter physics by itself, whose physical significance and applications still have many riches in store even forty years on, especially with the recent advent of mesoscopic and nanoscopic superconducting devices.

In this letter, we wish to report on new solutions to the LG equations which share some properties with usual Abrikosov configurations, but are nevertheless of quite a different nature and are possible because of the boundary conditions implied by mesoscopic samples of finite size. Rather than having the order parameter vanish along a line within the superconducting material as occurs for Abrikosov and giant magnetic vortices, these solutions are characterized by having the order parameter vanish on one or more *surfaces* each with the topology of a cylinder and enclosing one another in a concentric-like pattern with more or less constant spacing, while the maximal number of such surfaces is set by the size of the mesoscopic sample relative to the coherence length ξ of the material¹. Moreover, the volume defined by any pair of successive surfaces is threaded by a supercurrent running in a closed loop, parallel to and vanishing on these surfaces, with all these successive sheaths of currents having the same circular orientation. Finally, these configurations may also carry a non zero fluxoid quantum number, in which case they may be viewed as being bound to an ordinary giant vortex carrying that quantum of flux and lying at the center of the set of surfaces, the associated linear locus of the vanishing order parameter then threading along the central axis of the innermost surface. Because of this particular topology, we refer to these solutions as “annular vortices”² of order n and fluxoid L , n referring to the number of cylindrical surfaces on which the order parameter vanishes and L being the usual fluxoid quantum number. These solutions thus generalise the well known giant magnetic vortex ones which correspond to $n = 0$ and an arbitrary number L of Abrikosov vortices or anti-vortices lying on top of one another[3].

Since these configurations solve the LG equations, they define local extrema of the free energy, but it has not yet been established whether these are local minima rather than maxima or saddle points whatever the values for n and L . Irrespective of this issue of their dynamical stability in configuration space³, it is clear that for a fixed value of L , the free energy of these solutions increases with increasing n . Hence in practical physics terms, such configurations have only a finite thermodynamic lifetime which also still needs to be assessed.

Consequently, at this stage, it is not clear whether these annular vortices could be observed in mesoscopic devices, and how low a temperature would be required for that purpose. Nevertheless, from the quantum mechanical point view, annular vortices—even if dynamically unstable—provide for new contributions to the density of states of such systems, and should therefore affect specific physical properties of mesoscopic devices. One such instance could well be the behaviour in an external magnetic field of mesoscopic disks and annuli[5], whose maze of $(L, n = 0)$ energy curves in their phase diagram is then filled by many more such curves for $n \neq 0$, leading to a plethora of degeneracy crossings as a function of the applied field and thereby a dynamics of these devices possibly much richer still than that so far considered[6, 7].

In fact, having in mind eventual applications such as a new type of quantum detector for elementary particles and atoms, or as the basic qubit device for a quantum computer, these

¹A few weeks after these solutions had been found, we received Ref.[2] in which similar ones are discussed in the context of quantum optics, based on coupled non linear equations analogous to the LG ones.

²The idea of a vortex being associated to that of a current running in a closed loop.

³Note that the stability of the ordinary giant vortex solutions ($L, n = 0$) seems to have been settled only recently[4].

annular vortex solutions were found precisely when studying mesoscopic samples with a cylindrical and annular geometry, and assuming an infinite extent along their symmetry axis. In that case, in a plane perpendicular to that axis, the order parameter vanishes at the center when $L \neq 0$, while a series of almost regularly spaced n concentric circles of vanishing order parameter appear when $n \neq 0$ (in the case of an annulus, the same description applies with the central hole simply removed). This is the situation presented in this letter, but it is clear that for samples whose boundaries are of arbitrary shape, the same classes of solutions exist, obtained through continuous deformations of the perfectly symmetric ones constructed here, thereby leading to the general picture for these annular vortices as described above.

2. The LG equations. For time independent configurations in the presence of an external magnetic field \vec{B}_{ext} , in S.I. units the free energy of the system writes as[8]

$$E = \int_{(\infty)} d^3\vec{x} \frac{1}{2\mu_0} \left(\vec{B} - \vec{B}_{\text{ext}} \right)^2 + \int_{\Omega} d^3\vec{x} \frac{1}{2\mu_0} \left(\frac{\Phi_0}{2\pi\lambda^2} \right)^2 \lambda^2 \left\{ \left| \vec{\nabla}\psi - i\frac{q}{\hbar}\vec{A}\psi \right|^2 + \frac{1}{2\xi^2} [|\psi|^2 - 1]^2 - \frac{1}{2\xi^2} \right\} . \quad (1)$$

The notation used should be self-explanatory for most variables. The symbol Ω stands for the volume of the superconducting sample, $\Phi_0 = 2\pi\hbar/|q|$ —with $q = -2e$ (e being the positron charge)—is the usual unit of quantum of flux, while λ and ξ are the penetration and coherence lengths, respectively. However, we need to emphasize the fact that in this expression the order parameter $\psi(\vec{x})$ is normalised to its constant value in the bulk of the material in the absence of any magnetic field, and that the free energy is defined such as to vanish exactly at the normal-superconducting transition.

The above expression of the free energy makes it clear that it is the penetration length λ which weighs the relative contributions of the magnetic field energy—due to deviations of $\vec{B}(\vec{x})$ from the applied field \vec{B}_{ext} —and of the condensate energy—due to deviations of $|\psi(\vec{x})|$ from zero—, while it is the coherence length ξ which weighs the contributions to the condensate energy due to spatial inhomogeneities in $\psi(\vec{x})$ —through the covariantized gradient—relative to those due to deviations from the bulk value $|\psi| = 1$ —through the LG potential energy. This interplay of scales is made manifest in (1).

Extremizing the free energy leads to the LG equation

$$\lambda^2 \left[\vec{\nabla} - i\frac{q}{\hbar}\vec{A} \right]^2 \psi = \kappa^2 \psi [|\psi|^2 - 1] , \quad (2)$$

$\kappa = \lambda/\xi$ being the LG parameter, coupled the Maxwell equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{\text{em}} , \quad (3)$$

with the electromagnetic current density given by

$$\mu_0 \vec{J}_{\text{em}} = \frac{i}{2} \frac{\Phi_0}{2\pi\lambda^2} \left[\psi^* \left(\vec{\nabla}\psi - i\frac{q}{\hbar}\vec{A}\psi \right) - \left(\vec{\nabla}\psi^* + i\frac{q}{\hbar}\vec{A}\psi^* \right) \psi \right] , \quad \vec{\nabla} \cdot (\mu_0 \vec{J}_{\text{em}}) = 0 . \quad (4)$$

In particular, this latter relation implies the second London equation for the Meissner effect

$$\vec{\nabla} \times \left(\frac{\lambda^2}{|\psi|^2} \mu_0 \vec{J}_{\text{em}} \right) = -\vec{B} . \quad (5)$$

Let us now particularize these equations to axially symmetric configurations, the external homogeneous field \vec{B}_{ext} being also aligned with the symmetry axis \hat{e}_z . We shall specifically consider a sample either of cylindrical geometry with radius r_b (and $r_a = 0$) or of annular geometry with inner and outer radii r_a and r_b , respectively. Finally, in the present study

we assume the extension of the sample in the $\pm\hat{e}_z$ directions to be infinite. The following parametrisations then apply,

$$\vec{B} = B(r)\hat{e}_z, \quad \vec{A} = A(r)\hat{e}_\phi, \quad \mu_0\vec{J}_{\text{em}} = J(r)\hat{e}_\phi, \quad \psi(r, \phi) = f(r)e^{i\theta(\phi)}, \quad \rho(r) = |\psi(r, \phi)|^2 = f^2(r), \quad (6)$$

where (r, ϕ) define the polar coordinates in the plane perpendicular to the \hat{e}_z symmetry axis. In contradistinction to other works, an important point needs to be emphasized concerning the parametrisation used for the order parameter in terms of the two functions $f(r)$ and $\theta(\phi)$. The function $f(r)$ is of course real, *but of a sign which may be positive or negative as the case arises*, while the phase function $\theta(\phi)$ is assumed to be *continuous*. Usually, the function $f(r)$ is explicitly assumed to be the square-root of the (normalised) condensate density $\rho(r)$, $f(r) = \sqrt{\rho(r)}$, whose sign is thus always positive, with the important consequence that whenever the order parameter vanishes such that the gradient of $\sqrt{\rho}$ does not vanish as well, on the “other side of the zero” there is necessarily a discontinuous jump of $\pm\pi$ in the phase factor φ which is then used to parametrise the order parameter through $\psi = \sqrt{\rho}e^{i\varphi}$. There is of course no “other side of the zero” in the case of a giant vortex, but there is obviously one when ψ vanishes on a surface within the sample⁴. The general parametrisation of $\psi(\vec{x})$ advocated here allows for the specific possibility that $f(\vec{x})$ may change sign when crossing a zero of $\psi(\vec{x}) = f(\vec{x})e^{i\theta(\vec{x})}$, all in a continuous fashion for both functions $f(\vec{x})$ and $\theta(\vec{x})$ and their gradients. This point is obviously essential to our results.

To proceed further, let us choose to normalise quantities in terms of the penetration length and the associated unit of magnetic field $\Phi_0/(2\pi\lambda^2)$, by introducing the dimensionless variables,

$$u = \frac{r}{\lambda}, \quad b(u) = \frac{B(u)}{\Phi_0/(2\pi\lambda^2)}, \quad a(u) = \frac{1}{\lambda} \frac{A(u)}{\Phi_0/(2\pi\lambda^2)}, \quad g(u) = u \frac{q\lambda^3}{\hbar} \frac{1}{f^2(u)} J(u). \quad (7)$$

The coupled LG-Maxwell equations then reduce to[8]

$$\frac{1}{u} \frac{d}{du} \left[u \frac{d}{du} f(u) \right] = \frac{1}{u^2} f(u) g^2(u) - \kappa^2 f(u) [1 - f^2(u)] \quad , \quad u \frac{d}{du} \left[\frac{1}{u} \frac{d}{du} g(u) \right] = f^2(u) g(u) \quad , \quad (8)$$

while the magnetic field $B(r)$ is determined by the second London equation

$$b(u) = \frac{1}{u} \frac{d}{du} g(u) \quad . \quad (9)$$

Furthermore, a careful consideration of the boundary conditions (b.c.) of the problem, of the small u behaviour of solutions to these equations, of the flux threading the sample and of the relation (4) between $\mu_0\vec{J}_{\text{em}}$ and ψ , shows that the remaining quantities are determined to be[8]

$$\theta(\phi) = -L\phi + \theta_0 \quad , \quad ua(u) = g(u) + L \quad , \quad L = 0, \pm 1, \pm 2, \dots \quad , \quad (10)$$

where θ_0 is arbitrary and L is the usual fluxoid quantum number, thus corresponding to the order parameter $\psi(r, \phi) = f(r)e^{-iL\phi}e^{i\theta_0}$ characteristic of giant vortex configurations for $L \neq 0$. In particular, the total flux threading a disk of radius $u_a < u < u_b$ within the sample is

$$\Phi(u) = \Phi_0 [g(u) + L] = \Phi_0 ua(u) \quad , \quad u_a < u < u_b \quad . \quad (11)$$

The advantage of expressing the problem in these terms rather than those usual in the literature, is that only *gauge invariant* quantities still appear in the coupled equations to be solved, while the gauge variant ones, $\theta(\phi)$ and $ua(u)$, are explicitly known already.

⁴From that point of view, annular vortices are very much reminiscent of domain wall configurations in the spontaneously broken Yang-Mills theories of particle physics and cosmology, and could thus possibly be of interest in these other fields of physics as well, not to mention of course other coherent quantum systems in condensed matter physics.

The relevant b.c. still need to be specified. In the disk case surrounded by the vacuum or an insulating material, they are [6, 8]

$$\begin{aligned} \text{at } u = 0 & : g(u)|_{u=0} = -L & ; & \quad \frac{df(u)}{du}|_{u=0} = 0 \text{ if } L = 0 \text{ or } f(u)|_{u=0} = 0 \text{ if } L \neq 0 , \\ \text{at } u = u_b & : \frac{1}{u} \frac{dg(u)}{du}|_{u=u_b} = b_{\text{ext}} & ; & \quad \frac{df(u)}{du}|_{u=u_b} = 0 , \end{aligned} \quad (12)$$

while in the annulus case, we have [8]

$$\begin{aligned} \text{at } u = u_a & : \frac{1}{2} u_a \frac{dg(u)}{du}|_{u=u_a} = g(u_a) + L & ; & \quad \frac{df(u)}{du}|_{u=u_a} = 0 , \\ \text{at } u = u_b & : \frac{1}{u} \frac{dg(u)}{du}|_{u=u_b} = b_{\text{ext}} & ; & \quad \frac{df(u)}{du}|_{u=u_b} = 0 . \end{aligned} \quad (13)$$

Finally, it may be worth noting also that both these sets of b.c. as well as the coupled equations (8) for $f(u)$ and $g(u)$ derive from the variation of the following quantity (valid both for the disk with $u_a = 0$ and for the annulus with $u_a \neq 0$)

$$\begin{aligned} & \frac{1}{2} \int_{u_a}^{u_b} du u \left\{ \left[\frac{df}{du} \right]^2 + \left[\frac{1}{u} \frac{dg}{du} - b_{\text{ext}} \right]^2 + \frac{1}{u^2} f^2 g^2 + \frac{1}{2} \kappa^2 (1 - f^2)^2 - \frac{1}{2} \kappa^2 \right\} + \\ & + \frac{1}{4} u_a^2 \left[2 \frac{g(u_a) + L}{u_a^2} - b_{\text{ext}} \right]^2 = \frac{1}{2} \frac{2\mu_0}{2\pi\lambda^2} \left(\frac{2\pi\lambda^2}{\Phi_0} \right)^2 \mathcal{E} \equiv \mathcal{E}_{\text{norm}} , \end{aligned} \quad (14)$$

where \mathcal{E} denotes the free energy E per unit length in the \hat{e}_z direction of the sample.

3. Analytical insight. The resolution of the equations (8) requires a numerical approach, both because of the non linear character of these equations as well as the b.c. at finite values of r . Nevertheless, some analytical considerations may be developed, giving already the insight necessary to the construction of annular vortices and the actual demonstration of their existence.

The pair of coupled second order equations (8) requires a total of four b.c.. However, in order to determine a solution uniquely, these four conditions must be specified *at the same boundary*, thereby enabling the construction of the then unique solution by propagating these boundary values throughout the sample volume using the equations (8). In the present case, we do have a total of four such conditions, but split evenly between the two boundaries of the sample at u_a ($u_a = 0$ in the disk case) and u_b . In order to construct solutions, one needs to add two free b.c. at either boundary, propagate the solution within the sample and then adjust the two free conditions in order to meet the two conditions specified at the other boundary. Consequently, this procedure may generate more than a single solution.

To be specific, consider first the disk case and introduce the parametrisation

$$f(u) = u^{|L|} \tilde{f}(u) ; \quad f_0 = \tilde{f}(u)|_{u=0} , \quad g_0 = \frac{1}{2u} \frac{dg(u)}{du}|_{u=0} = \frac{1}{2} b(u)|_{u=0} . \quad (15)$$

From the equations (8), it follows that $\tilde{f}(u)$ and $g(u)$ are both even functions of u , while the two boundary conditions at $u = 0$ in (12) are then also satisfied. Thus, the parameters f_0 and g_0 correspond to the two additional free boundary values at $u = 0$ in order to construct a unique configuration through (8), with in particular g_0 related to the value b_{ext} of the external field. From a straightforward linearisation of the LG equation, one then finds exactly

$$\lim_{f_0, g_0 \rightarrow 0} \frac{1}{f_0} f(u) = |L|! \left(\frac{2}{\kappa} \right)^{|L|} J_{|L|}(\kappa u) , \quad \lim_{f_0, g_0 \rightarrow 0} g(u) = -L , \quad \lim_{f_0, g_0 \rightarrow 0} b_{\text{ext}} = 0 , \quad (16)$$

where $J_{|L|}(x)$ is the Bessel function of the first kind. In other words, precisely at the normal-superconducting transition in a vanishing external field, there is in fact lurking in the still waters of $f(u) = 0$ an oscillatory pattern just waiting to emerge as soon as $f_0 \neq 0$, namely as soon as the superconducting phase is reached. Thus for f_0 arbitrary close to zero but not vanishing, by continuity there exists a solution to (8) and the b.c. (12) at $u = 0$ for which $f(u)$ passes through

a succession of zero values while alternating in sign. In the limit ($f_0 = 0, g_0 = 0$), these values are the zeros of the Bessel function $J_{|L|}(\kappa u)$. To a good approximation, their spacing in the sample is such that $\Delta r \approx \pi \xi$, while the number of those lying within the radius of the sample defines an upper bound on the maximal number $n_{\max}(L)$ of annular vortices possible, differing from it by at most one unit since the b.c. (12) at $u = u_b$ still need to be solved, and whose value $n_{\max}(L)$ thus decreases as $|L|$ increases.

This remark also leads to the following picture. Having fixed $g_0 = 0$, let f_0 increase (or decrease, the sign is physically irrelevant) from $f_0 = 0$, and generate configurations for $(f(u), g(u))$ using (8). Since f_0 sets the value for $f(u)$ (when $L = 0$) or for its $|L|$ -th order derivative (when $L \neq 0$) at $u = 0$, and because of the negative driving force $-\kappa^2 f(u) [1 - f^2(u)]$ (for $|f(u)| \leq 1$) in the LG equation, the initial oscillatory pattern of the Bessel function is then pushed further and further outwards, with the position of the innermost zero of $f(u)$ moving out while the spacing Δr between successive zeros retains more or less the above constant value (due to the condensate rigidity set by ξ). Consequently, more and more zeros of $f(u)$ leave the sample until configurations with no zeros left within it are reached. Moreover, throughout the increase of f_0 from the null value, there is a value $f_0^{(n)}$ to be found within each of the successive intervals in f_0 which are associated to the loss of a zero in $f(u)$ from the sample, such that the b.c. $df(u_b)/du = 0$ in (12) is also satisfied. Correspondingly, the solution for $g(u)$ then sets the associated value for b_{ext} through the second b.c. at $u = u_b$ in (12). Hence in conclusion, with L fixed and $g_0 = 0$, this demonstrates the existence of (L, n) annular vortex solutions with $n = 0, 1, \dots, n_{\max}(L)$, characterized by having the order parameter vanish on concentric cylinders when $n \neq 0$ and on the symmetry axis when $L \neq 0$.

In the $(b_{\text{ext}}, \mathcal{E}_{\text{norm}})$ phase diagram, the above procedure determines a certain curve starting from the origin⁵. Along that curve, there thus appears in succession a series of points associated to the annular vortex solutions with a value of n decreasing from $n_{\max}(L)$ to $n = 0$. Each of these points thus also defines a local extremum of the free energy (14) (which may then take either a negative or a positive value), and in fact the numerical analysis[8] shows that these are local minima in the parameter space (f_0, g_0) (nevertheless, this is a far cry from having established dynamical stability for these solutions within the entire configuration space).

Finally, the complete set of solutions to (8) with the b.c. (12) is generated for whatever values of b_{ext} by letting now g_0 run up and down its parameter space starting from $g_0 = 0$, thereby generating energy curves in the $(b_{\text{ext}}, \mathcal{E}_{\text{norm}})$ phase diagram associated to each of the (L, n) annular vortex solutions. This is to be done for each of the $(f_0^{(n)}, g_0 = 0)$ solutions constructed above, by adjusting appropriately the value of f_0 for each new value of g_0 . To each such solution, there corresponds a specific value for b_{ext} set by the b.c. in (12). However, this correspondence between g_0 and b_{ext} is not necessarily one-to-one, as is indeed demonstrated by the numerical analysis. Thus, there may exist more than one solution of fixed (L, n) for some values of the external field, a fact responsible for hysteresis and switching phenomena between states of different L values in a varying field, associated to the crossing points of the negative energy curves of (L, n) vortices in the $(b_{\text{ext}}, \mathcal{E}_{\text{norm}})$ phase diagram.

The Maxwell equation in (8) shows that the magnetic field $b(u)$ is stationnary at the zeros of $f(u)$ (there is no Meissner effect at those points) and that it is monotonous in between zeros of $g(u)$. Moreover, since $g(u) \sim uJ(u)/f^2(u)$ is finite at the nodes of $f(u)$, the supercurrent $J(u)$ must also vanish at these locations while it remains of constant sign for as long as $g(u)$ does not cross the null value. In particular, when L and b_{ext} are of opposite sign, $g(u)$ does not vanish within the sample, so that all the successive supercurrent vortices retain the same circular orientation.

⁵A series expansion analysis of the solutions shows that in the case $L = 0$, this curve runs exactly along the negative $\mathcal{E}_{\text{norm}}$ axis, in which case $g_0 = 0$ always implies $b_{\text{ext}} = 0$.

The same considerations apply in the case of the annular geometry, the two free b.c. to be adjusted in a likewise fashion being then the values of $f(u)$ and $g(u)$ at $u = u_a$. What replaces then the result (16) is a linear combination of the Bessel functions of the first and second kind, $J_{|L|}(\kappa u)$ and $N_{|L|}(\kappa u)$, while otherwise all the previous considerations remain applicable, with the obvious provision that the total number n of zeros in $f(u)$ is reduced as compared to the disk case because of the inner hole of radius r_a .

Hence, the above analytical considerations, though not providing explicit solutions, demonstrate beyond any doubt the existence of the annular vortex solutions with the characteristics described in the Introduction. However, the important issue of their dynamical stability (*i.e.* the existence of negative modes of the quadratic operators in (8)) remains open. A remark possibly of interest in this respect, to be detailed elsewhere[8], is the following. It is well known[3] that for the critical value $\kappa = 1/\sqrt{2}$, the ordinary giant vortex solutions ($L, n = 0$) in the infinite dimensional plane saturate the Bogomol'nyi bound[9] for the free energy by satisfying first-order differential equations from which (8) then follows. However, given rather the b.c. (12) and (13) associated to a sample of finite radial extent, it may be shown that these first-order differential equations are incompatible with these b.c., so that the solutions ($L, n \neq 0$) cannot be considered as being Bogomol'nyi type topological configurations. One may argue that the solutions ($L, n = 0$) are only a slight deformation of the Bogomol'nyi configurations implied by the b.c., but this remark does not apply to the ($L, n \neq 0$) solutions which are certainly not degenerate in energy and thus cannot all saturate the same Bogomol'nyi bound. Indeed, as soon as $f(u)$ crosses the null value at some radius, from there on at larger radii the function $f(u)$ keeps vanishing at almost regular intervals in an oscillatory pattern, so that the free energy of such configurations keeps increasing with increasing values of n since each successive supercurrent vortex contributes some additional amount of kinetic energy. This is the physical reason why annular vortex solutions were never found through formal mathematical analyses of the space of solutions to the LG equation in the infinite plane[3].

4. Numerical solutions. The above considerations are of course explicitly confirmed through the numerical resolution of the equations (8) with the b.c. (12) and (13). By lack of space, only a small—though representative—sample of examples is presented[8] in the disk case (those for the annulus are very much similar). The method followed is the one described above using the free parameters (f_0, g_0) and a 4th order Runge-Kutta integration of the differential equations.

Fig.1 displays the ($b_{\text{ext}}, \mathcal{E}_{\text{norm}}$) phase diagram for a choice of radius $u_b = r_b/\lambda = 5$ and LG parameter $\kappa = 1$. Even though only the curves for $L = 0, \pm 1, \pm 2, \pm 3$ are shown, the ($L, n = 0$) giant vortex configurations follow of course the usual pattern. Solutions with $n = 1$ exist only for $L = 0$. Indeed, for $L \neq 0$, because of the central volume taken up by the magnetic vortex at the center (see Fig.2), the position of the first zero of $f(u)$ is so much pushed outwards that it leaves the sample for this choice of radius. For larger values of $\kappa u_b = r_b/\xi$, $n = 1$ solutions also exist for $L \neq 0$, in which case a series of energy curves similar to those shown in Fig.1 for the $n = 0$ solutions also appear for the $n = 1$ solutions (and so on for still larger values of n , provided κu_b is large enough; the choice $\kappa u_b = 5$ is made in order that Fig.1 still looks simple enough to the eye). Clearly, the crossings of all such curves render the dynamics of such superconducting devices of large enough radii still far richer and more complicated than that provided already by the usual $n = 0$ configurations[7].

Another feature of Fig.1 worthwhile to emphasize is the caustic-like structure in the energy curves around their crossing points with the zero energy axis. The appearance of these cusps stems from the method used to generate configurations. Solutions move along these curves as the value of g_0 varies, so that the relation between g_0 and b_{ext} is not necessarily one-to-one for some intervals in these quantities. Moreover, solutions then also move into the positive energy domain, until they meet a return point back towards the zero energy axis which they reach sometimes after having moved back into the negative energy domain as well. Quite clearly, this

specific feature displayed by our approach[8] should prove to be important to an understanding of the hysteresis phenomena which exist around the critical fields for the normal-superconducting phase transitions to the (L, n) solutions (even though not apparent on Fig.1, $n \neq 0$ solutions also possess such caustic behaviour), whose physical properties are determined by the dynamic and thermodynamic decay rates between the normal and these different superconducting configurations. These rates are function of the applied field B_{ext} and the temperature T (note that $\lambda(T)$ and $\xi(T)$ also depend on the latter parameter), as well as the value and especially the sign of the rate of change in the applied field.

Fig.2 displays all the configurations which exist with $L = 0, 1$ for a disk of radius $u_b = 11$ and LG parameter $\kappa = 1$, submitted to an external field of $b_{\text{ext}} = 0.05$ (for $\lambda = 50$ nm, this corresponds to $B_{\text{ext}} \approx 66$ Gauss). The values of $f(u)$, $b(u)$ and $f^2(u)g(u)/u = q\lambda^3 J(u)/\hbar \sim -J(u)$ are shown for the $(L = 0; n = 0, 1, 2, 3)$ and $(L = 1; n = 0, 1, 2)$ solutions. In the latter case, the solution with $n = 3$ has disappeared because of the space taken up by the central magnetic vortex, which thus pushes outwards the whole train of zeros of the $(L = 0, n = 3)$ solution. Note also that for this disk with $u_b = 11$ and $\kappa = 1$, the values of each of the solutions $n = 0, 1, 2$ coincide almost exactly beyond $u > 0.4u_b$ when comparing the $L = 0$ and $L = 1$ configurations, since the effects of the $L = 1$ magnetic vortex remain confined within a radius of a few penetration lengths λ . Further properties of these configurations have been discussed in general terms already, in particular the fact that the magnetic field $b(u)$ is stationary at nodes of the order parameter $\psi(r, \phi)$, where the supercurrent $J(u)$ then also vanishes, thereby enabling further penetration of the external field within the superconducting sample and a partial anti-screening of the Meissner effect for $n \neq 0$ annular vortices as compared to the usual $n = 0$ configurations. These few examples thus confirm exactly the results advocated on basis of the general analytical considerations developed previously.

5. Conclusions. This letter has demonstrated the existence of annular vortex solutions to the Landau-Ginzburg equations in mesoscopic superconductors. These configurations include and generalise the well known Abrikosov and giant magnetic vortex ones, are characterized by properties described above, and exist only thanks to the finite spatial extent of mesoscopic devices. Important issues remain open obviously, most of them raised by the eventual experimental observation of direct or indirect physical effects related to the existence in actual samples of such annular vortices, either as single entities or organised into regular or irregular ensembles. Dynamic and thermodynamic stability is obviously essential for their direct observation, and remains to be investigated. Irrespective of their stability, these configurations should have indirect manifestations since they enrich the phase diagram of mesoscopic superconducting disks and loops in an external magnetic field in ways still to be explored. Besides hysteresis phenomena in such systems to which our approach should be applicable, annular vortices also provide for a new mechanism for the switching between states of different fluxoid quantum number, which—since these solutions manifestly share the cylindrical symmetry of the sample—is in direct competition with the specific mechanism unravelled in Ref.[10]. Static and dynamic properties of mesoscopic superconducting devices should thus be affected by the existence of such annular vortices, in ways to be assessed explicitly and whose conclusions could potentially be of importance when having specific practical applications of such devices in mind. We plan to report on progress on some of these issues in later work.

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Figure Captions

Figure 1: The normalised free energy $\mathcal{E}_{\text{norm}}$ of (14) as a function of the normalised applied field b_{ext} for a disk with $u_b = 5$ and $\kappa = 1$. Displayed are the $n = 0$ giant vortex configurations with $L = -3, -2, -1, 0, 1, 2, 3$ from left to right in that order (thin parabolic curves), while the $n = 1$ solution exists for $L = 0$ only (thick line).

Figure 2: The $(L = 0; n = 0, 1, 2, 3)$ and $(L = 1; n = 0, 1, 2)$ configurations for a disk with $u_b = 11$, $\kappa = 1$ and $b_{\text{ext}} = 0.05$. Displayed in that order from left to right are the values for $f(u)$, $b(u)$ and $f^2(u)g(u)/u = q\lambda^3 J(u)/\hbar \sim -J(u)$ as functions of the variable $x = u/u_b = r/r_b$, $0 \leq x \leq 1$. The top (resp. bottom) panels correspond to $L = 0$ (resp. $L = 1$). The $f(u)$ values for the $(L = 0, n = 0)$ solution coincide almost exactly with $f(u) = 1$ and cannot be distinguished from unity on these graphs (one has $f(u_b) = 0.9995072$ and $f(u_b) = 0.9995076$ for $(L = 0, n = 0)$ and $(L = 1, n = 0)$, respectively).

